



## Euclidean versus Non-Euclidean Geometry

### Lesson Synopsis:

In this lesson, students explore different geometric systems and compare and contrast the structure of those systems with that of Euclidean geometry. In particular, students explore taxicab geometry and geometric concepts such as distance, area, and perimeter. Students solve real-world problems in a taxicab setting and compare the implications of those solutions in a Euclidean setting. In addition, students develop an understanding of the necessity of other geometric systems as they examine the geometric implications of the sphere and concave surfaces. Students then compare and contrast Euclidean geometry to spherical and hyperbolic geometry.

### TEKS:

**G.1** *Geometric structure. The student understands the structure of, and relationships within, an axiomatic system.*

**G.1C** Compare and contrast the structures and implications of Euclidean and non-Euclidean geometries.

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## GETTING READY FOR INSTRUCTION

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### Performance Indicator(s):

- Compare and contrast Euclidean and taxicab geometry by writing a brief summary about the structures and implications of the geometric definitions and formulas. Compare and contrast spherical and hyperbolic geometry with Euclidean geometry by writing a brief summary about the structures and implications of geometric properties. (G.1C)

**ELPS** ELPS: 1E, 1H, 2E, 2I, 3H, 4I, 5G

### Key Understandings and Guiding Questions:

- There are other systems of geometry besides Euclidean in which structures behave differently and may be better applied to real-world situations.
  - How does the measurement of distance in taxicab geometry differ from Euclidean geometry?
  - How does the boundary of the region equal distance from a point differ in taxicab geometry and Euclidean geometry?
  - Why does the Euclidean parallel postulate not hold true in spherical geometry?
  - Why does the Euclidean parallel postulate not hold true in hyperbolic geometry?
  - Why does the sum of the angles of a triangle differ in spherical and Euclidean geometry?
  - Why does the sum of the angles of a triangle differ in hyperbolic and Euclidean geometry?

### Vocabulary of Instruction:

- |                      |                       |                |
|----------------------|-----------------------|----------------|
| • Euclidean geometry | • hyperbolic geometry | • great circle |
| • taxicab geometry   | • latitude            | • antipodal    |
| • spherical geometry | • longitude           | • lunes        |

### Materials:

- |                    |   |                   |
|--------------------|---|-------------------|
| • highlighters     | • string                                      | • plastic spheres |
| • erasable markers | • concave disk (inside of a cut plastic ball) | • protractors     |
| • scissors         |   | • rulers          |

### Resources:

-  **STATE RESOURCES**
  - **Mathematics TEKS Toolkit:** Clarifying Activity/Lesson,/Assessments  
<http://www.utdanacenter.org/mathtoolkit/index.php>

— **TEXTTEAMS: High School Geometry: Supporting TEKS and TAKS**

I – Structure; 2.0 Student Activity – Taxicab Geometry, 2.1, Act. 1 (Taxicab Geometry); III – Triangles; 2.0 Student Activity – Spherical Geometry, 2.1, Act. 1 (Student Activity: Spherical Geometry)

## Advance Preparation:

- Handout: **As the Crow Flies** (1 per student)
- Handout: **Anytown, U.S.A.** (1 per student)
- Handout: **When Is a Circle a Square?** (1 per student)
- Handout: **Which Firehouse?** (1 per student)
- Handout: **Spherical and Hyperbolic Geometry** (1 per student)
- Handout: **A Deeper Look at Spherical Geometry** (1 per student)
- Handout: **Spherical Geometry** (1 per student)
- Handout: **What Space Are You Living In?** (1 per student)

## Background Information:

In order to fulfill the requirements of the student expectation for this lesson, students should be well versed in previous topics of Euclidean geometry. Although this lesson only begins to examine the implications of taxicab, spherical, and hyperbolic geometry, students will benefit from a sound understanding of the Euclidean concepts of circles, perpendicular bisectors, area, perimeter, parallel lines, and sum of angles of a triangle.

## GETTING READY FOR INSTRUCTION SUPPLEMENTAL PLANNING DOCUMENT

Instructors are encouraged to supplement, and substitute resources, materials, and activities to differentiate instruction to address the needs of learners. The Exemplar Lessons are one approach to teaching and reaching the Performance Indicators and Specificity in the Instructional Focus Document for this unit. A Microsoft Word template for this planning document is located at [www.cscope.us/sup\\_plan\\_temp.doc](http://www.cscope.us/sup_plan_temp.doc). If a supplement is created electronically, users are encouraged to upload the document to their Lesson Plans as a Lesson Plan Resource in your district Curriculum Developer site for future reference.

## INSTRUCTIONAL PROCEDURES

### Instructional Procedures

#### ENGAGE

1. Distribute the handout: **As the Crow Flies** to each student.
2. Have students read the passage and facilitate a discussion around the questions that follow.  
Facilitation questions:
  - **In the passage you read, what is meant by the expression “as the crow flies”?** *The expression “as the crow flies” implies a point to point distance or a straight line distance between two points.*
  - **In the passage you read, how can there be two different distances to the same place? Explain.** *The “as the crow flies” distance is the point to point or linear distance between two points. The actual travel distance is the second type of distance or distance measure.*
  - **Revisit the Ruler Postulate. What do your answers to questions 1 and 2 reveal about the Ruler Postulate? What does this mean about geometry?** *The previous answers contradict the Ruler Postulate which states that the distance between two points is unique. Since there can be different ways to measure distances, there must be different geometric systems.*

### Notes for Teacher

**NOTE:** 1 Day = 50 minutes  
**Suggested Day 1 (½ day)**

#### MATERIALS

- Handout: **As the Crow Flies** (1 per student)

#### TEACHER NOTE

The purpose of the ENGAGE is to expose students to a situation that apparently contradicts the Euclidean notion of distance. Facilitating a discussion of different ways to measure distance sets the stage for students to recognize that just as there are different distance measures, there are different geometric systems in which other properties or measures such as distance may or may not behave as in a Euclidean setting.



#### STATE RESOURCES

**Mathematics TEKS Toolkit:** Clarifying Activity/Lesson/Assessment may be used to reinforce these concepts or used as alternate activities.

## Instructional Procedures

### EXPLORE/EXPLAIN 1

Day 1 (1/2 day)

1. Distribute the handout: **Anytown, U.S.A.** to each student.
2. Give each student a highlighter or colored pencil. Have students complete problems #1-2 on p. 1 of the handout in pairs. Have volunteers share their diagrams and discuss the first page in whole group.

Day 2

3. Go over problems #3-4 on p. 2 of **Anytown, U.S.A.** in whole group discussion. Have students work with a partner to complete the definitions at the bottom of the page. Have volunteers share definitions in whole group discussion.
4. Have students continue to work with their partner to complete problems #5-11 on pp. 3-5. These may be completed as homework, if necessary.

Day 3

5. Debrief answers on problems #5-11 of **Anytown, U.S.A.** in whole group discussion.
6. Have students work with their partner to complete #12-15.
7. When students have completed the handout, debrief with the following questions.

Facilitation questions:

- **Why do you suppose the geometry of Anytown, U.S.A. is referred to as taxicab geometry?** *The lines in the space are much like the paths that taxicabs would drive in a city; hence the name, taxicab geometry.*
- **What method did you use to find the distance between two points in Anytown, U.S.A.?** *The taxicab distance is the sum of the vertical and horizontal distances between the points.*
- **How is the distance measure in taxicab geometry different from Euclidean distance measure?** *Euclidean distance measure allows for the diagonal distance between two points, whereas taxicab geometry distance allows only for vertical and horizontal measure.*
- **What formula can be used to calculate taxicab distances given a coordinate system and two points?** *Path*  
$$\text{Distance} = |x_1 - x_2| + |y_1 - y_2|.$$
- **Given a point in a taxicab space, can you describe the set of points that are equidistant from the given point? How is this situation different from that of a Euclidean space?** *In a taxicab space, the set of points is a square, while in a Euclidean space, the set of points is a circle.*

### ELABORATE 1

1. Distribute the handout: **When Is a Circle a Square?** to each student.
2. Have students complete the activity with a partner.

## Notes for Teacher

**Suggested Day 1-3 (2 ½ days)**

### MATERIALS

- Handout: **Anytown, U.S.A.** (1 per student)
- highlighters or markers

### TEACHER NOTE

The purpose of the EXPLORE 1 is to allow students to explore a taxicab space. Students will develop an understanding of distance and other traditional geometric concepts in a taxicab geometry setting as they complete the activity.



### STATE RESOURCES

**TEXTTEAMS: High School Geometry: Supporting TEKS and TAKS**

I – Structure; 2.0 Student Activity – Taxicab Geometry, 2.1, Act. 1 (Taxicab Geometry) may be used to reinforce these concepts or used as alternate activities.

**Suggested Day 4**

### MATERIALS

- Handout: **When Is a Circle a Square?** (1 per student)

### TEACHER NOTE

The purpose of this activity is to compare and contrast the concept of a circle in Euclidean geometry and taxicab geometry.

### TEACHER NOTE

As students complete **When Is a Circle a Square?**, they compare the Euclidean

Instructional Procedures	Notes for Teacher
	<p>notion of a circle to that of taxicab geometry by developing area and perimeter (circumference) models from collected data.</p>
ELABORATE 2	Suggested Day 5-6 (2 days)
<ol style="list-style-type: none"> <li>1. Distribute the handout: <b>Which Firehouse?</b> to each student.</li> <li>2. Have students complete the activity with a partner or in groups.</li> <li>3. Debrief the activity <b>Which Firehouse?</b> with appropriate questions.</li> </ol> <p>Facilitation questions:</p> <ul style="list-style-type: none"> <li>• <b>How would you describe the set of all points that are equidistant from two given points in both a Euclidean space and a taxicab space as outlined in the Which Firehouse? activity?</b> <i>In the activity, the Euclidean solution is simply the perpendicular bisector of the segment formed by the two given points. The taxicab solution is the intersections along the diagonal from the corner of 8<sup>th</sup> street and Pine to the corner of 4<sup>th</sup> Street and Birch. Intersections on the boundary of, or within the region to the North of Pine Street and to the East of 8<sup>th</sup> Street are equidistant to the fire stations, and intersections on the boundary of, or within the region to the South of Birch Street and to the West of 4<sup>th</sup> Street are equidistant to the fire stations.</i></li> <li>• <b>Can you think of a situation in which the Euclidean solution and the taxicab solution would be the same?</b> <i>Yes. If the two given points have the same x-coordinate or the same y-coordinate, then the taxicab solution is the same as the Euclidean solution, simply a perpendicular bisector of the segment formed by the two given points.</i></li> </ul>	<p><b>MATERIALS</b></p> <ul style="list-style-type: none"> <li>• Handout: <b>Which Firehouse?</b> (1 per student)</li> <li>• highlighters or markers</li> </ul> <p><b>TEACHER NOTE</b></p> <p>The purpose of this activity is to compare and contrast the concept of equidistant points in Euclidean geometry and taxicab geometry. By modeling the problem in a real-world setting, students can identify the value of other geometric systems such as taxicab geometry.</p>
EXPLORE/EXPLAIN 2	SUGGESTED DAYS 7-8 (1 ½ days)
<ol style="list-style-type: none"> <li>1. Distribute the handout: <b>Spherical and Hyperbolic Geometry</b> to each student.</li> <li>2. Use a plastic sphere and concave disk as models and go over the handout in whole group discussion.</li> <li>3. Put students into pairs or small groups. Distribute the handout: <b>A Deeper Look at Spherical Geometry</b> to each student. Make sure each group has a plastic sphere, erasable marker, string, protractor, ruler, and scissors to cut the string.</li> <li>4. Direct students to read pp. 1-2 and use the provided materials to explore and/or verify concepts from the reading.</li> </ol>	<p><b>MATERIALS</b></p> <ul style="list-style-type: none"> <li>• Handout: <b>Spherical and Hyperbolic Geometry</b> (1 per student)</li> <li>• Handout: <b>A Deeper Look at Spherical Geometry</b> (1 per student)</li> <li>• concave disk</li> <li>• plastic spheres that can be marked on and erased (beach balls)</li> <li>• erasable markers</li> <li>• scissors</li> <li>• string</li> <li>• protractors</li> <li>• rulers</li> </ul> <p><b>TEACHER NOTE</b></p> <p>The purpose of this activity is to allow students to investigate spherical and hyperbolic geometry and contrast them with Euclidean geometry.</p>
ELABORATE 3	Suggested Day 8 (½ day)
<ol style="list-style-type: none"> <li>1. Distribute the handout: <b>Spherical Geometry</b> to each student.</li> <li>2. Have students use the previous handouts to answer the questions.</li> </ol>	<p><b>MATERIALS</b></p> <ul style="list-style-type: none"> <li>• Handout: <b>Spherical Geometry</b> (1 per student)</li> </ul>

## Instructional Procedures

## Notes for Teacher

### TEACHER NOTE

The purpose of this activity is to allow students to apply their understanding of spheres to answer questions and describe characteristics of the Earth.



### STATE RESOURCES

#### TEXTEAMS: High School Geometry: Supporting TEKS and TAKS

III – Triangles; 2.0 Student Activity – Spherical Geometry, 2.1, Act. 1 (Student Activity: Spherical Geometry) may be used to reinforce these concepts or used as alternate activities.

## EVALUATE

1. Distribute copies of handout: **What Space Are You Living In?** to students.
2. Have students complete the activity in order to assess their ability to accurately compare and contrast Taxicab and Spherical geometries with Euclidean geometry.

### Suggested Day 9

#### MATERIALS

- Handout: **What Space Are You Living In?**

### TEACHER NOTE

The prompts contained within the EVALUATE are directed to relate back to experiences from the previous phases of the lesson cycle and yet be open-ended. Therefore, the teacher should carefully read student responses in order to determine a student's understanding of Euclidean and non-Euclidean geometries.

## As the Crow Flies **KEY**

Coach Jackson was so impressed with your baseball field layout from the previous unit that he asked you to help lay out the field for the season. After class yesterday, he gave you money to purchase two steel tape measures to lay out the field but you forgot to purchase them. Coach Ryan tells you that you had better hurry and get those tape measures or face the consequences, a visit with Coach Ruth, the athletic director who has earned the title “the Sultan of Swat.” With a couple of hours to spare, you set out to purchase the tapes. On the way to the town of Wrigley, you stop by Cobb’s Peach Orchard and ask the attendant, “How far is it to Joltin’ Joe’s Hardware?” Ty, the attendant and quite the baseball fanatic himself, says, “Well, as the crow flies, it’s only about 3 miles, but for you it will be 5 miles. Just follow this road into town and Joe’s is just off of Elm Street. You can’t miss it. Hey...tell Hammering Hank up there at Joe’s that I asked about him.” As you continue on your way, you begin to wonder what the old timer meant by “as the crow flies” and how there could be two distances to Joltin’ Joe’s Hardware.

1. In the passage above, what is meant by the expression “as the crow flies”?  
The expression “as the crow flies” implies a point to point distance or a straight line distance between two points.
2. In the passage above, how can there be two different distances to the same place? Explain.  
The “as the crow flies” distance is the point to point or linear distance between two points.  
The actual travel distance is the second type of distance measure.
3. Revisit the Ruler Postulate. What do your answers to questions 1 and 2 reveal about the Ruler Postulate? What does this mean about geometry?  
The previous answers contradict the Ruler Postulate which states that the distance between two points is unique. Since there can be different ways to measure distances, there must be different geometric systems.

## As the Crow Flies

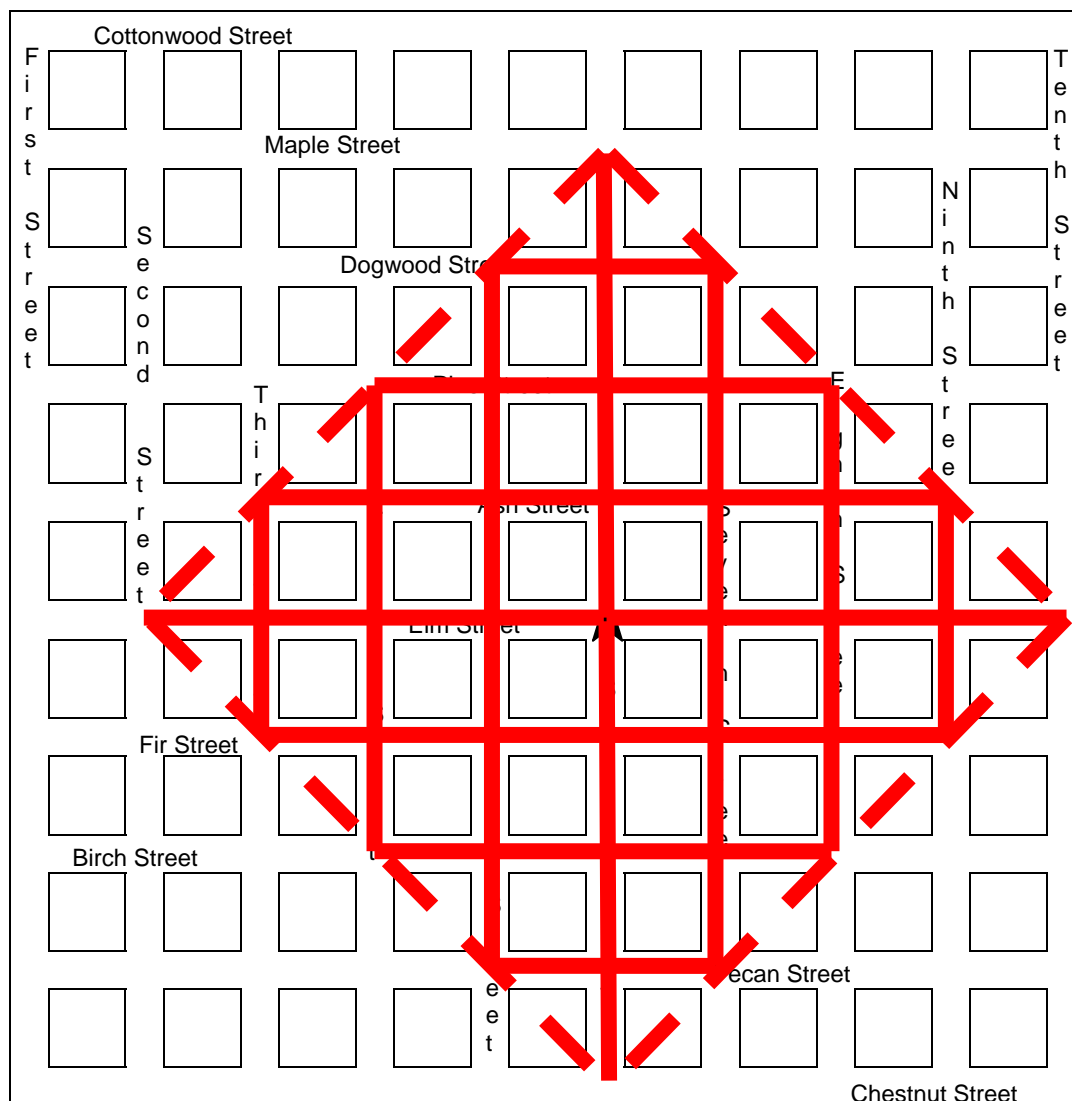
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1. In the passage above, what is meant by the expression “as the crow flies”?
2. In the passage above, how can there be two different distances to the same place? Explain.
3. Revisit the Ruler Postulate. What do your answers to questions 1 and 2 reveal about the Ruler Postulate? What does this mean about geometry?

## Anytown, U.S.A. (pp. 1 of 6) **KEY**

The map below represents the downtown area of Anytown U.S.A. Office Supplies, Inc. is putting in a store at the corner of Sixth Street and Elm. Two weeks before the office opens, the owners would like to send fliers to all offices within four blocks of the new store.

- On the map, use a highlighter to shade in the street paths that represent four blocks from the store. Use another color of marker to draw dashed boundary lines to represent the region receiving fliers.



- What is the shape formed to represent the region equidistant in blocks from a given point?  
The shape of the region is a square.



## Anytown, U.S.A. (pp. 2 of 6) **KEY**

Let's explore the geometry of Anytown, U.S.A. against what we already understand about high school or Euclidean geometry.

3. Euclidean geometry assumptions represent relationships between the undefined terms: **point**, **line**, and **plane**. The following assumptions are sometimes called axioms or postulates and are used to prove theorems.
- If two distinct lines intersect, their intersection is a point.
  - If there is a line and a point not on the line, there is exactly one line through the point parallel to the given line.
  - If two distinct planes intersect, their intersection is a line.

The system of geometry used in Anytown U.S.A. is actually called taxicab geometry. Why do you think this name was selected?

**The lines in the space are much like the paths that taxicabs would drive in a city.**

4. In taxicab geometry, the undefined terms are point, street, and line. The following are assumptions that can be made in taxicab geometry. If you have other assumptions, add them to the list.
- Streets have no thickness.
  - Consecutive streets are equal distance apart.
  - Streets are parallel or perpendicular.
  - Blocks contain buildings that cannot be walked through.
  - Lines can be drawn through buildings
  - Path distance can only be measured along streets.
  - Imaginary distances can be measured along any lines.

Use the undefined terms and assumptions for taxicab geometry to define the following. If you can define other terms, add them to the list.

- Intersection  
**An intersection is the point at which two streets meet.**
- Line segment  
**A line segment is part of a line.**
- Diagonal line segment  
**A diagonal line segment is a line segment drawn between non-consecutive intersections.**
- Block  
**A block is the length of a line segment between any two consecutive intersections.**
- Boundary line  
**A boundary line is a line enclosing a region.**
- Path distance  
**The path distance is the least number of blocks between intersections or points.**
- Imaginary distance  
**Imaginary distance is the length of line segments between non-consecutive intersections.**

## Anytown, U.S.A. (pp. 3 of 6) **KEY**

Use your understanding of taxicab geometry from questions 3 and 4 and the diagram of Anytown, U.S.A. from question 1 to answer the following.

5. In Anytown, U.S.A., Jeff and Melissa work at an office on the corner of Second Street and Maple Street. Lee Ann and Bryan work at an office on the corner of Eighth Street and Chestnut Street. Your office is at Third Street and Fir Street.

a. Whose office is closest to your own?

**Jeff and Melissa**

b. How did you determine your answer?

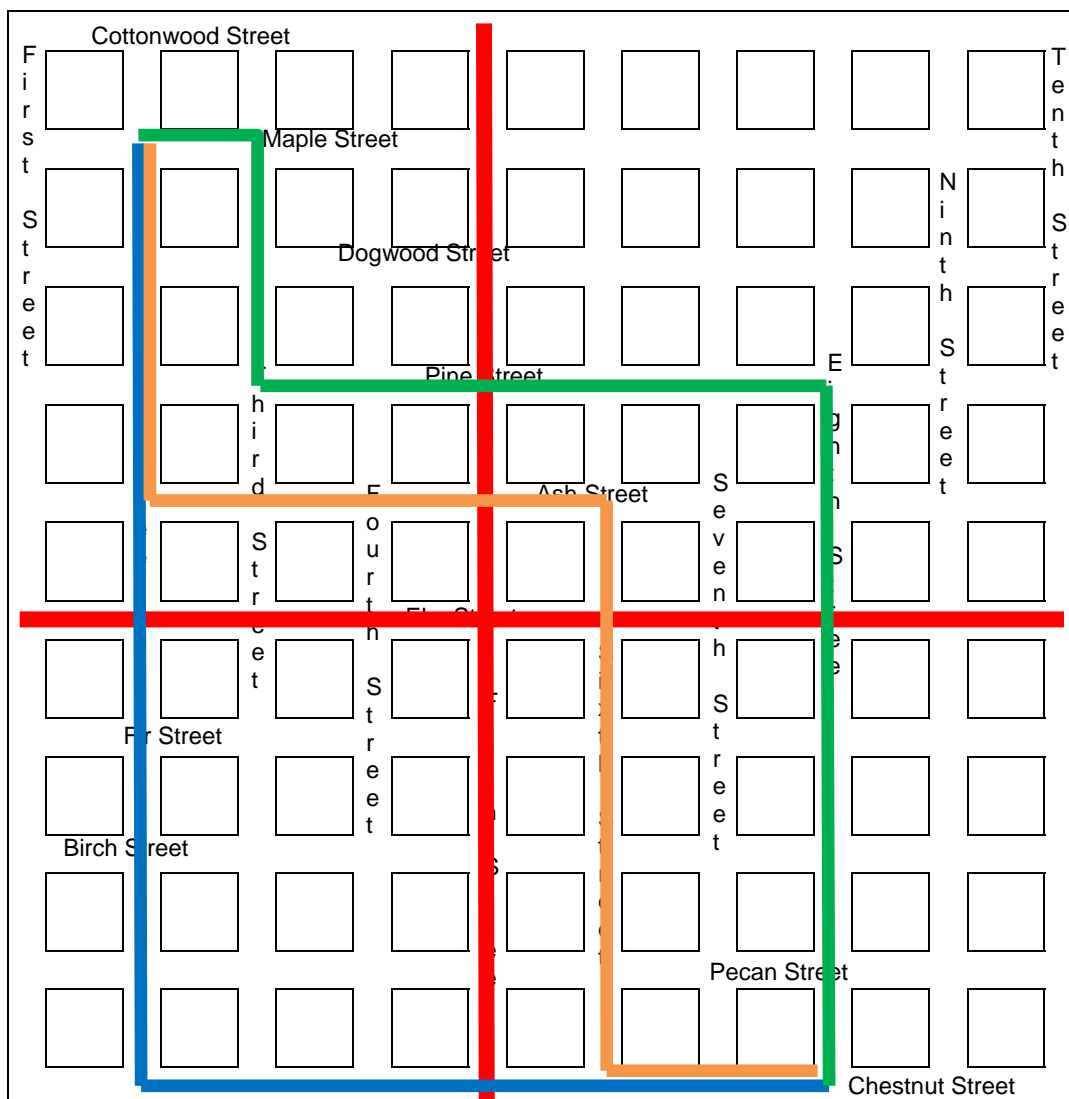
**The shortest distance from Jeff and Melissa's office is 6 blocks, while the shortest distance from LeeAnn and Bryan's office is 8 blocks.**

6. Write a definition for the shortest path distance between two points.

**The shortest path distance between two points is the least number of blocks between intersections or points.**

## Anytown, U.S.A. (pp. 4 of 6) **KEY**

Use a highlighter to shade in Fifth Street and Elm Street to represent the axes of a coordinate plane.



7. Jeff and Melissa work at an office on the corner of Second Street and Maple Street. What are the coordinates of their office?

**(-3, 4)**

8. Lee Ann and Bryan work at an office on the corner of Eighth Street and Chestnut Street. What are the coordinates of their office?

**(3, -4)**

## Anytown, U.S.A. (pp. 5 of 6) **KEY**

9. Create three different shortest paths from one office to the other using a different color marker for each path. Find the total vertical distance traveled in each path. What is true about each of the vertical distances?

The vertical distances are the same regardless of the shortest path taken; 8 blocks.

10. Using the same paths from question 9, find the total horizontal distance traveled in each path. What is true about each of the horizontal distances?

The horizontal distances are the same regardless of the shortest path taken; 6 blocks.

11. Based on your answers to questions 3 and 4, summarize how you would find the taxicab distance between the two offices. Does your summary agree with your answer to question 6?

The Taxicab distance between the two offices is the sum of the horizontal and vertical distances, or in this case 6 blocks + 8 blocks = 14 blocks traveled. Yes, the shortest path distance between two points in a Taxicab space is the sum of the horizontal and vertical distances.

12. Given the coordinates of two locations,  $(x_1, y_1)$  and  $(x_2, y_2)$ , how can you express the taxicab distance symbolically?

$$\text{Path distance} = |x_1 - x_2| + |y_1 - y_2|$$

13. Use your formula from question 12 to find the distance between  $(2, 3)$  and  $(-2, 1)$ . Show your work below.

$$\begin{aligned}\text{Path distance} &= |2 - (-2)| + |3 - 1| \\ &= 4 + 2 \\ &= 6\end{aligned}$$

## Anytown, U.S.A. (pp. 6 of 6) **KEY**

14. Find the Euclidean distance between the two locations given in question 7.

Since distance is positive,

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$d = \sqrt{(2 - (-2))^2 + (3 - 1)^2}$$

$$d = \sqrt{(4)^2 + (2)^2}$$

$$d = \sqrt{16 + 4}$$

$$d = \sqrt{20}$$

$$d = 2\sqrt{5} \approx 4.47$$

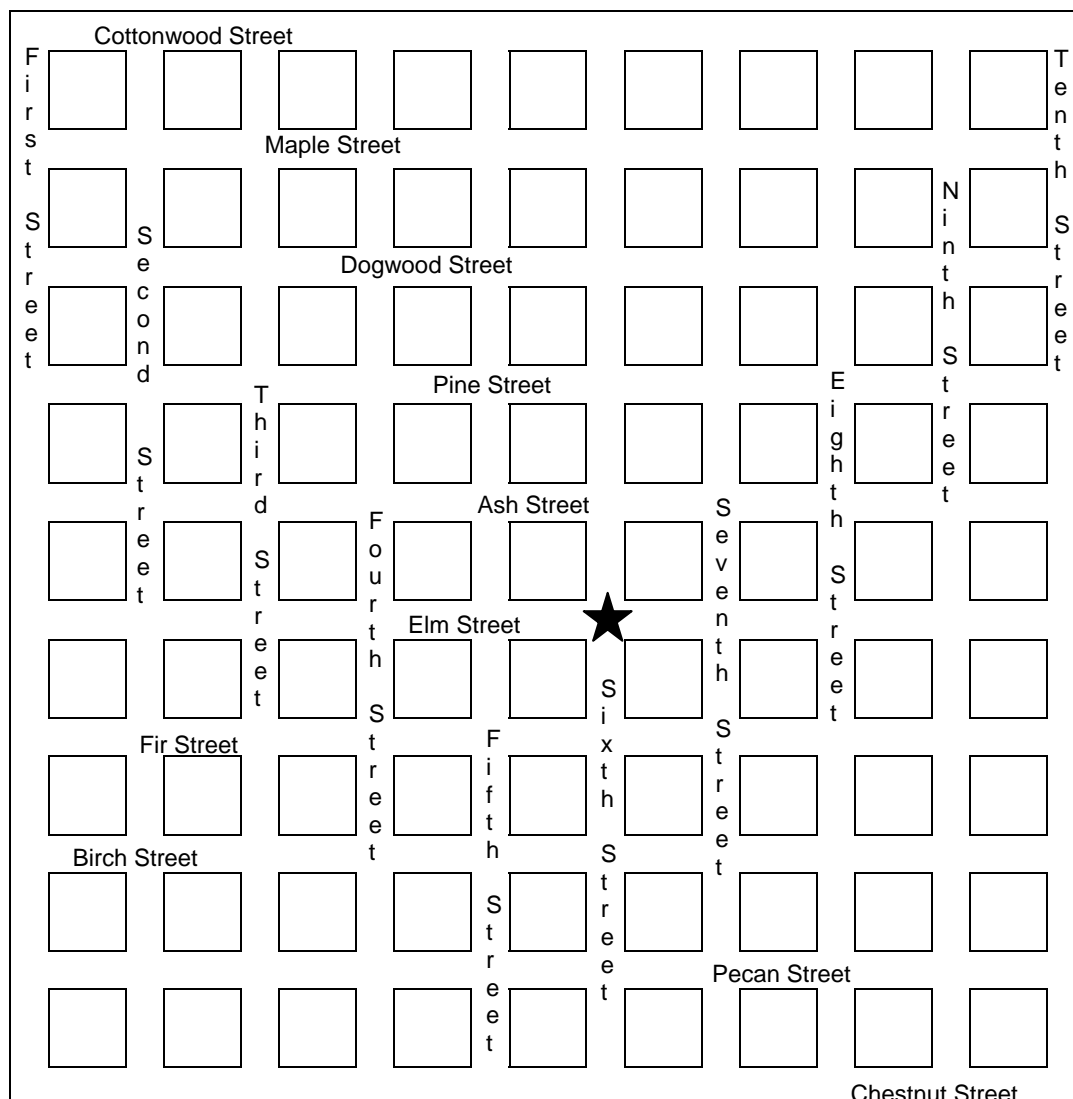
15. What conjectures can you make about Euclidean distances and taxicab distances?

In general, given two points the taxicab distance between the two points is larger than the Euclidean distance. An exception to this occurs if the two points have the same x-coordinate or the same y-coordinate. In such cases, the distances are the same.

## Anytown, U.S.A. (pp. 1 of 6)

The map below represents the downtown area of Anytown U.S.A. Office Supplies, Inc. is putting in a store at the corner of Sixth Street and Elm. Two weeks before the office opens, the owners would like to send fliers to all offices within four blocks of the new store.

- On the map, use a highlighter to shade in the street paths that represent four blocks from the store. Use another color of marker to draw dashed boundary lines to represent the region receiving fliers.



- What is the shape formed to represent the region equidistant in blocks from a given point?

## Anytown, U.S.A. (pp. 2 of 6)

Let's explore the geometry of Anytown, U.S.A. against what we already understand about high school or Euclidean geometry.

3. Euclidean geometry assumptions represent relationships between the undefined terms: **point**, **line**, and **plane**. The following assumptions are sometimes called axioms or postulates and are used to prove theorems.
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The system of geometry used in Anytown U.S.A. is actually called *taxicab* geometry. Why do you think this name was selected?

4. In taxicab geometry, the undefined terms are point, street, and line. The following are assumptions that can be made in taxicab geometry. If you have other assumptions, add them to the list.
- Streets have no thickness.
  - Consecutive streets are equal distance apart.
  - Streets are parallel or perpendicular.
  - Blocks contain buildings that cannot be walked through.
  - Lines can be drawn through buildings
  - Path distance can only be measured along streets.
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Use the undefined terms and assumptions for taxicab geometry to define the following. If you can define other terms, add them to the list.

- Intersection
- Line segment
- Diagonal line segment
- Block
- Boundary line
- Path distance
- Imaginary distance

## **Anytown, U.S.A.** (pp. 3 of 6)

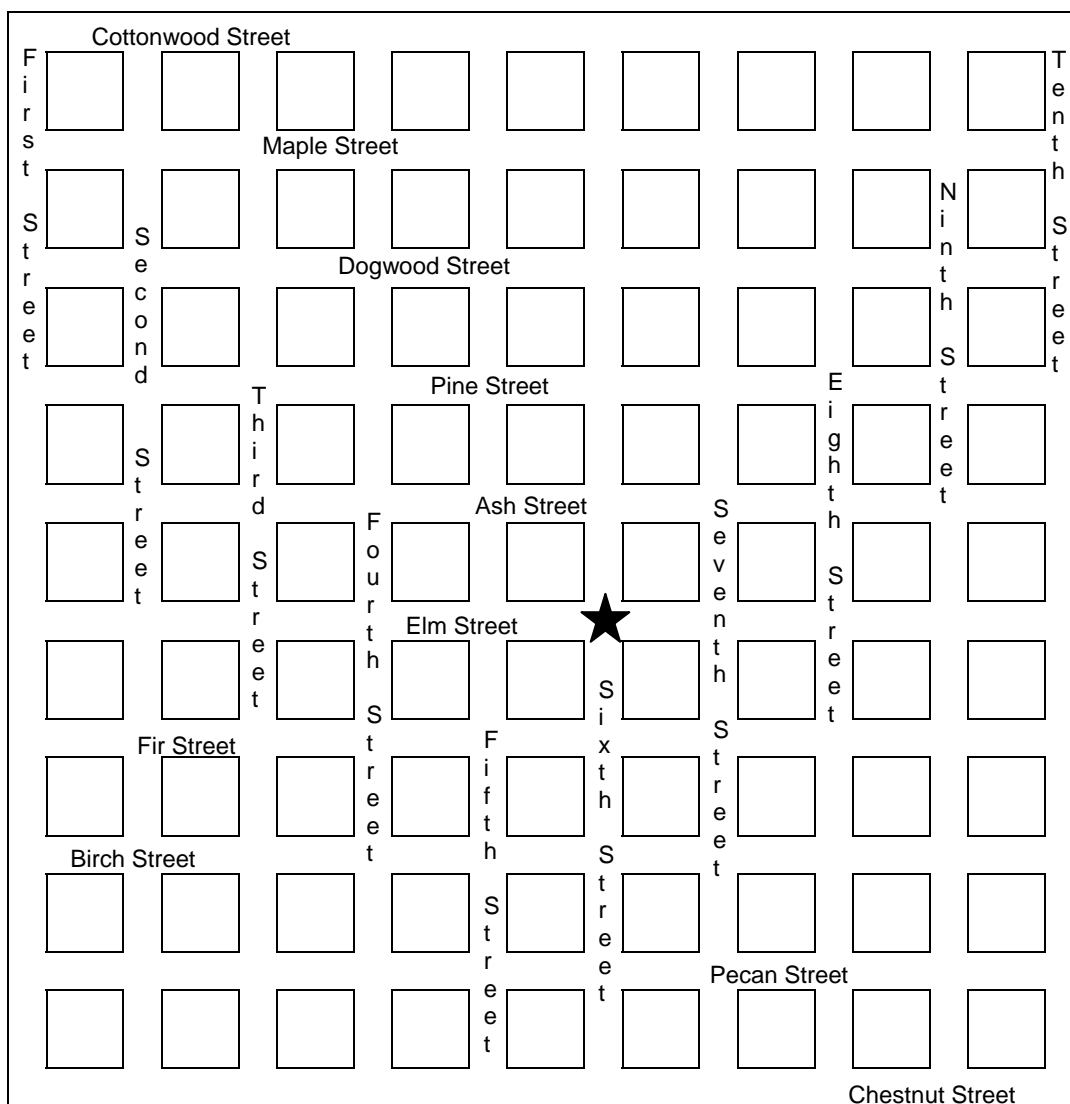
Use your understanding of taxicab geometry from questions 3 and 4 and the diagram of Anytown, U.S.A. from question 1 to answer the following.

5. In Anytown, U.S.A., Jeff and Melissa work at an office on the corner of Second Street and Maple Street. Lee Ann and Bryan work at an office on the corner of Eighth Street and Chestnut Street. Your office is at Third Street and Fir Street.
  - a. Whose office is closest to your own?
  - b. How did you determine your answer?
6. Write a definition for the shortest path distance between two points.



## Anytown, U.S.A. (pp. 4 of 6)

Use a highlighter to shade in Fifth Street and Elm Street to represent the axes of a coordinate plane.



7. Jeff and Melissa work at an office on the corner of Second Street and Maple Street. What are the coordinates of their office?
  
8. Lee Ann and Bryan work at an office on the corner of Eighth Street and Chestnut Street. What are the coordinates of their office?

## Anytown, U.S.A. (pp. 5 of 6)

9. Create three different shortest paths from one office to the other using a different color marker for each path. Find the total vertical distance traveled in each path. What is true about each of the vertical distances?
10. Using the same paths from question 9, find the total horizontal distance traveled in each path. What is true about each of the horizontal distances?
11. Based on your answers to questions 3 and 4, summarize how you would find the taxicab distance between the two offices. Does your summary agree with your answer to question 6?
12. Given the coordinates of two locations,  $(x_1, y_1)$  and  $(x_2, y_2)$ , how can you express the taxicab distance symbolically?
13. Use your formula from question 12 to find the distance between  $(2, 3)$  and  $(-2, 1)$ . Show your work below.

## **Anytown, U.S.A.** (pp. 6 of 6)

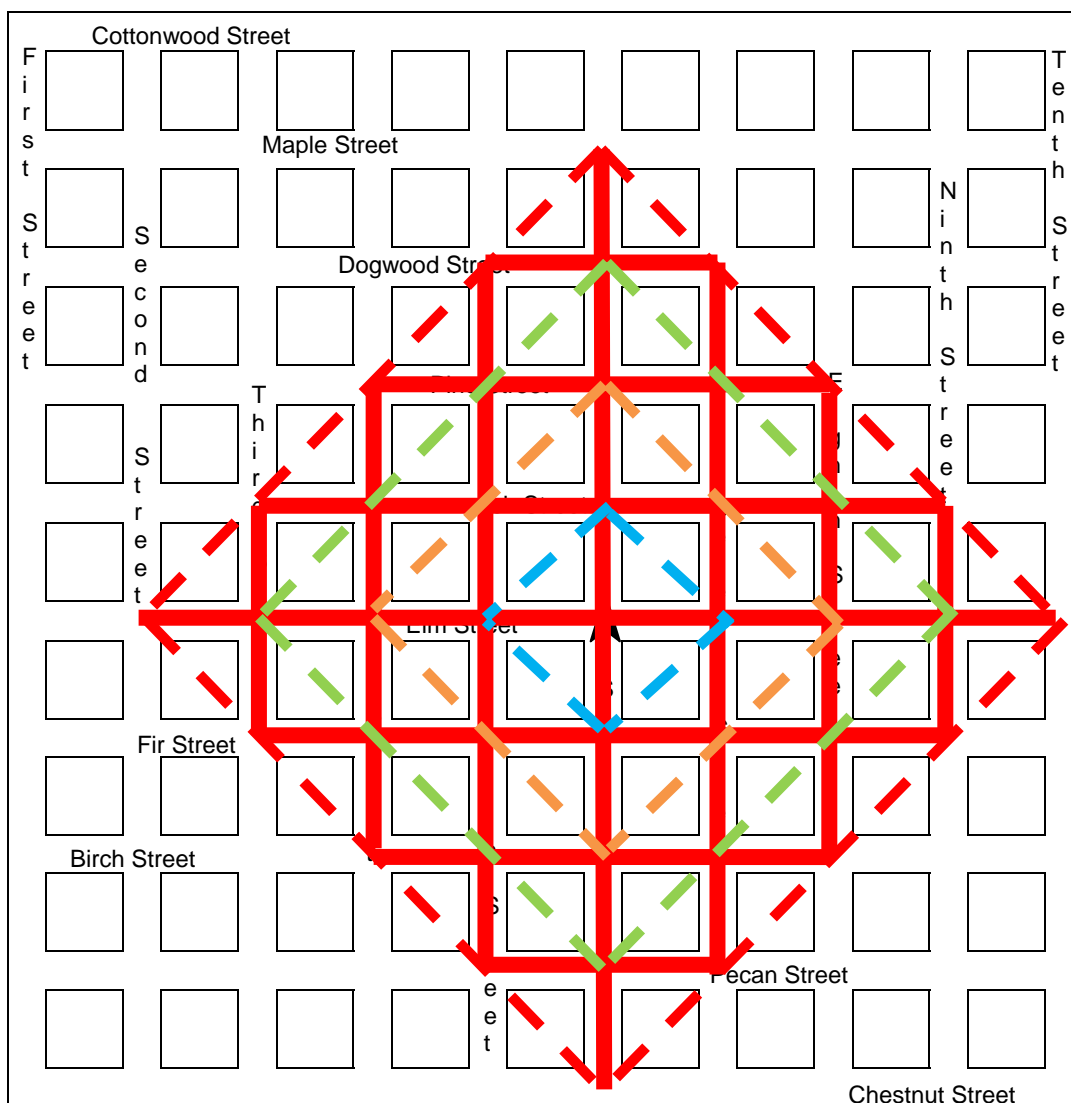
14. Find the Euclidean distance between the two locations given in question 7.

15. What conjectures can you make about Euclidean distances and taxicab distances?

## When Is a Circle a Square? (pp. 1 of 2) **KEY**

Remember that in your initial visit to Anytown, U.S.A., you were asked to distribute flyers from Joltin' Joe's Hardware to all offices within a certain number of blocks from the store. Let's examine this situation a little more closely.

On the map below, Joltin' Joe's Hardware is located at the corner of 6<sup>th</sup> Street and Elm. Use a highlighter to shade in the street paths that are one block from the store, two blocks from the store, three blocks from the store, and so on. Draw the boundary line for each region using a different colored pencil or marker. As you complete each region, take a moment to record your findings in the chart on the next page.



## When Is a Circle a Square? (pp. 2 of 2) **KEY**

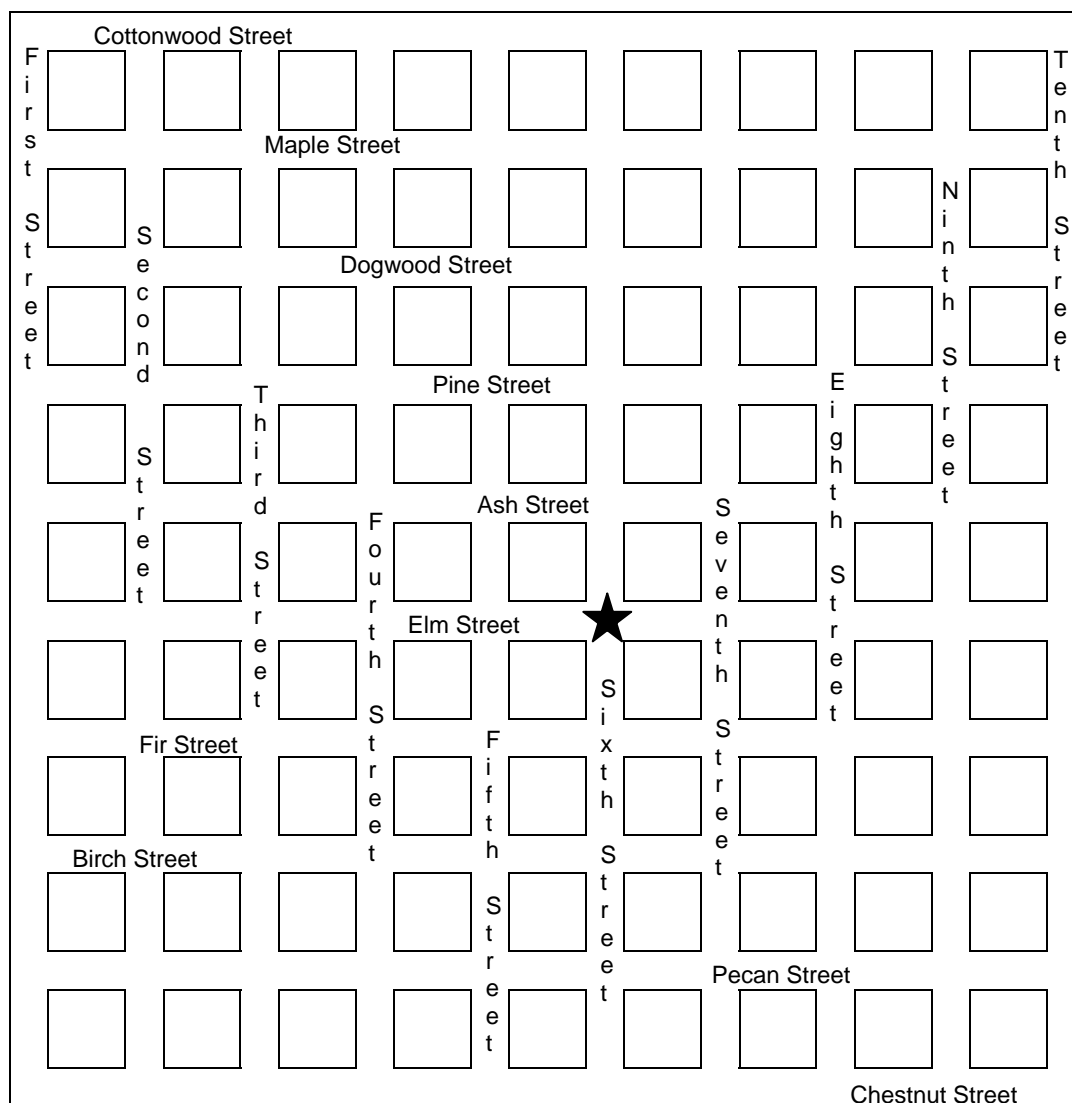
Number of Blocks from Joe's	Perimeter of Region	Area of Region
1	$4\sqrt{2}$	2
2	$8\sqrt{2}$	8
3	$12\sqrt{2}$	18
4	$16\sqrt{2}$	32
x	$4x\sqrt{2}$	$2x^2$

- As you collected data for the table above, you found the set of all points some given distance from Joltin' Joe's Hardware store in a taxicab space. Describe the regions created given a distance of one block unit, two block units, three block units and four block units.  
**In each case, the resulting region is a square.**
- Describe the method you used for finding the perimeter and area of each region.  
**Student answers may vary considerably. For example, for area some students may use the formula for the area of a rhombus since the diagonals are easily discernable given the distance from the store. Easier still and more intuitive is to relate the regions to  $45^\circ$ - $45^\circ$ - $90^\circ$  triangles and simply count the lengths of the hypotenuses for the perimeter, and count the number of square block units for area.**
- If you have not already done so, determine an expression for the perimeter and area of a taxicab region that is x units from the store. Record your answers below.  
**Area =  $2x^2$ , and Perimeter =  $4x\sqrt{2}$ .**
- Suppose the store is in a Euclidean space represented by point J. What would the resulting regions look like if you found the set of all points in a plane that are 1 unit from the store, 2 units from the store, 3 units from the store, and 4 units from the store?  
**The resulting regions would be concentric circles of radii 1, 2, 3, and 4.**

## When Is a Circle a Square? (pp. 1 of 2)

Remember that in your initial visit to Anytown, U.S.A., you were asked to distribute flyers from Joltin' Joe's Hardware to all offices within a certain number of blocks from the store. Let's examine this situation a little more closely.

On the map below, Joltin' Joe's Hardware is located at the corner of 6<sup>th</sup> Street and Elm. Use a highlighter to shade in the street paths that are one block from the store, two blocks from the store, three blocks from the store, and so on. Draw the boundary line for each region using a different colored pencil or marker. As you complete each region, take a moment to record your findings in the chart on the next page.



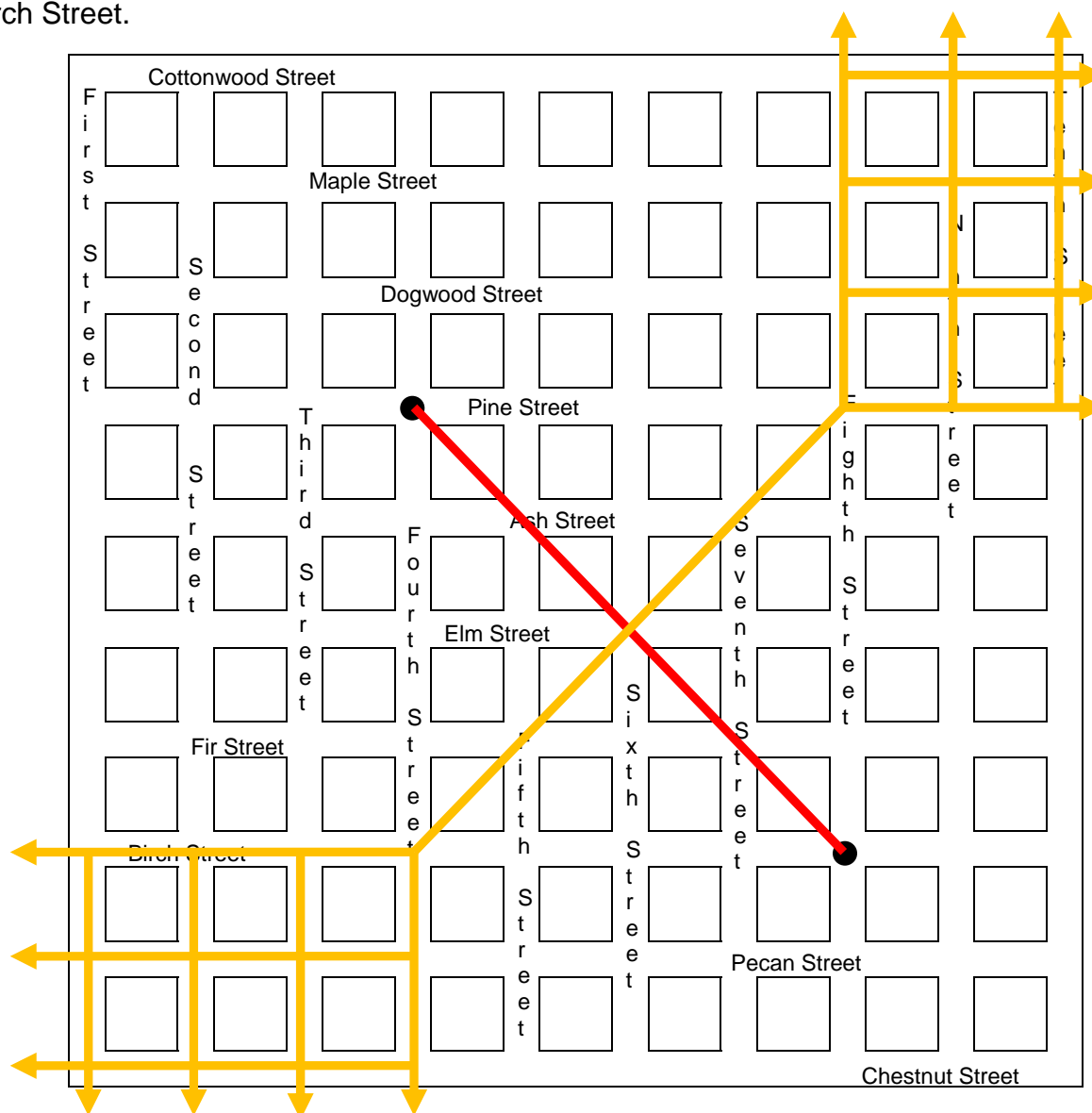
## When Is a Circle a Square? (pp. 2 of 2)

Number of Blocks from Joe's	Perimeter of Region	Area of Region
1		
2		
3		
4		
x		

- As you collected data for the table above, you found the set of all points some given distance from Joltin' Joe's Hardware store in a taxicab space. Describe the regions created given a distance of one block unit, two block units, three block units and four block units.
- Describe the method you used for finding the perimeter and area of each region.
- If you have not already done so, determine an expression for the perimeter and area of a taxicab region that is  $x$  units from the store. Record your answers below.
- Suppose the store is in a Euclidean space represented by point J. What would the resulting regions look like if you found the set of all points in a plane that are 1 unit from the store, 2 units from the store, 3 units from the store, and 4 units from the store?

## Which Firehouse? (pp. 1 of 2) **KEY**

Let's revisit Anytown, U.S.A. Consider the diagram below. The city's fire department has just constructed stations at the corner of Fourth Street and Pine Street, and at the corner of Eighth Street and Birch Street.



Suppose the Fire Chief is assigning zones of response to each station under his jurisdiction.

1. Use a colored marker to draw a line segment from one fire station to the other. How far is it from one fire station to the other fire station along the segment (Euclidean distance)? What is the path distance (taxicab distance)?

The Euclidean distance is  $4\sqrt{2} \approx 5.66$ .

The taxicab distance is 8.



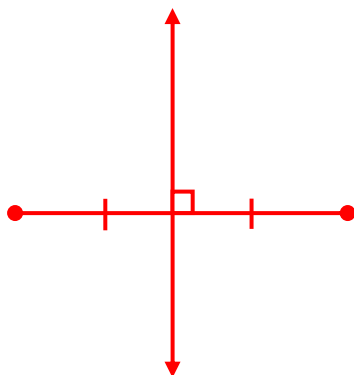
## Which Firehouse? (pp. 2 of 2) **KEY**

2. Use a different color marker to draw the set of all locations that are equidistant from each of the fire stations. Describe the set of points that are equidistant from the fire stations.

Intersections along the diagonal from the corner of Eighth Street and Pine to the corner of Fourth Street and Birch are equidistant to the fire stations. Intersections on the boundary of, or within the region to the North of Pine Street and to the East of Eighth Street are equidistant to the fire stations. Intersections on the boundary of, or within the region to the South of Birch Street and to the West of Fourth Street are equidistant to the fire stations.

3. Suppose you found the set of all points in a plane equidistant from the endpoints of a segment in Euclidean geometry. What would the result be? Sketch a diagram below.

In a Euclidean space, the set of all points in a plane equidistant from the endpoints of a segment is the perpendicular bisector of the segment.

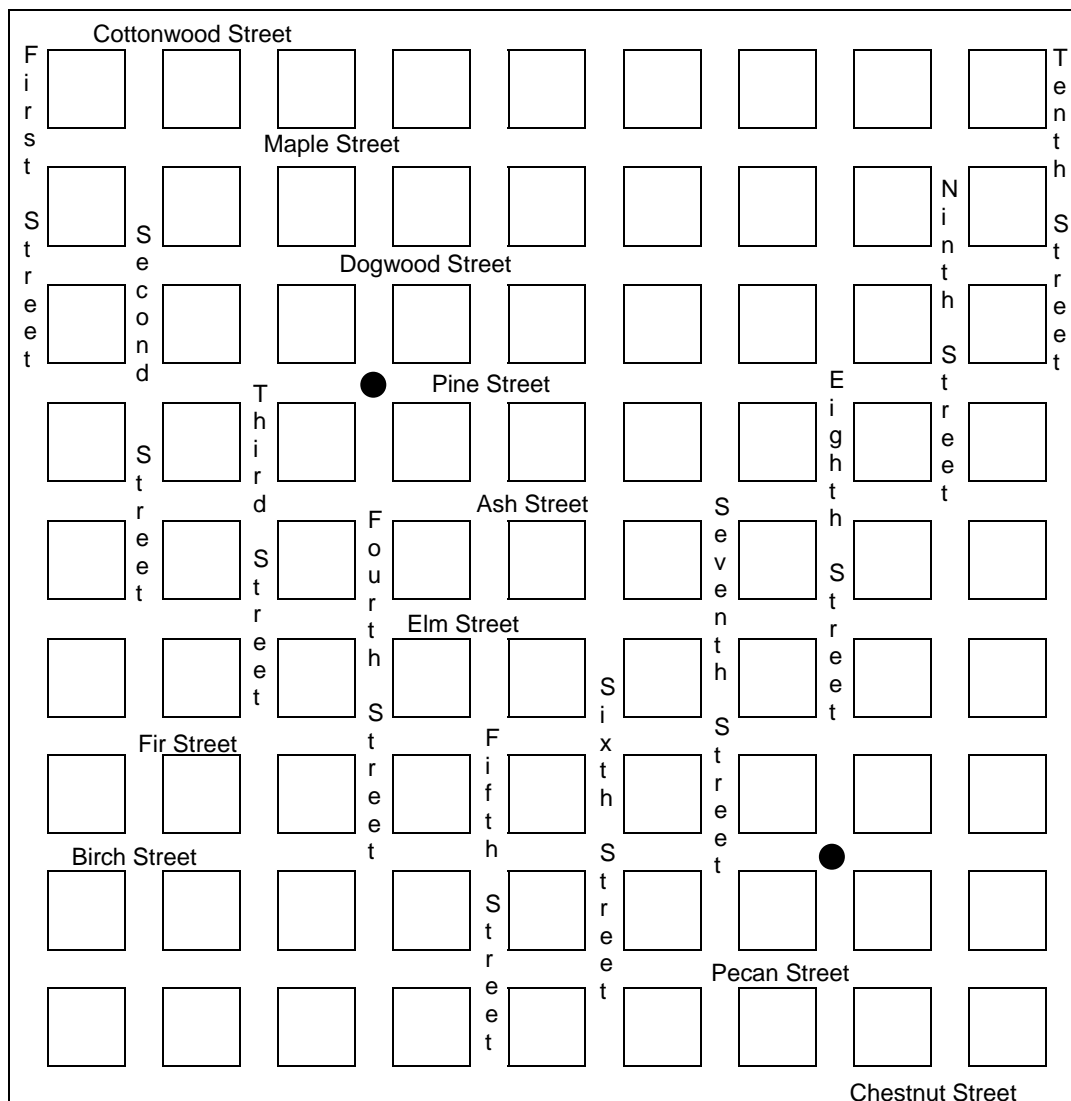


4. How is your solution to question 2 different from finding the set of all points in a plane equidistant from the endpoints of a segment in Euclidean geometry? Explain.

The solution in the Taxicab space includes a two dimensional region of points, whereas the solution in Euclidean geometry is simply a line or one-dimensional.

## Which Firehouse? (pp. 1 of 2)

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## Spherical and Hyperbolic Geometry (pp. 1 of 2)

The famous mathematician Euclid is credited with being the first person to describe geometry. One of the assumptions in Euclidean geometry is the Parallel Postulate which states that through a given point not on a line, there is exactly one line passing through the point parallel to the given line. Geometry in which the Parallel Postulate is assumed is known as "Euclidean" or "flat" geometry.

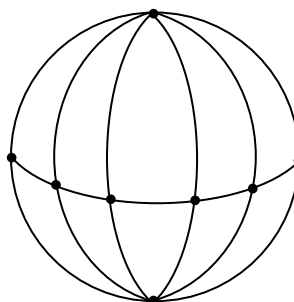
This postulate was so complicated that many felt it should be proved rather than taken as an assumption. In the two thousand years following Euclid, many mathematicians, professional and amateur, tried and failed to prove it.

If the Parallel Postulate is considered false, then one of the following assumptions must be considered true.

- ▶ Assumption 1: Through a given point not on a line, there are no lines parallel to the given line.
- ▶ Assumption 2: Through a given point not on a line, there is more than one line parallel to the given line.

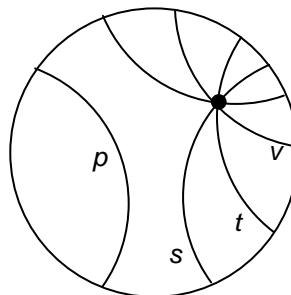
Assumption 1 applies to spherical geometry. Assume that the following figure is a sphere. Lines are represented by curves on the great circles of the sphere.

This is a sphere and in spherical geometry, great circles represent lines. Note that all great circles intersect. Therefore there are no parallel "lines".



Assumption 2 applies to hyperbolic geometry. Assume that the following figure is a concave disk. Lines appear to be curves on the surface of the disk, although if flattened into a plane, they would be linear.

This is not a sphere, but is the inside of a curved disk. Note that  $p$  is parallel to  $s$ ,  $t$ , and  $v$ , since they do not intersect. There are an infinite number of curves that could be drawn through the point and not intersect  $p$ .



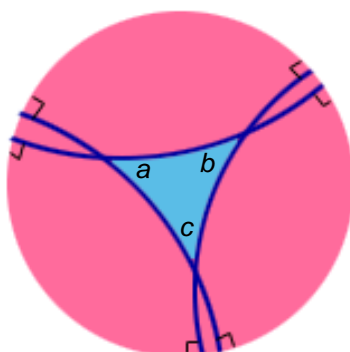
## Spherical and Hyperbolic Geometry (pp. 2 of 2)

Another postulate in Euclidean geometry states that the sum of the angles of a triangle equals  $180^\circ$ .

If this postulate is considered false, then one of the following assumptions must be considered true.

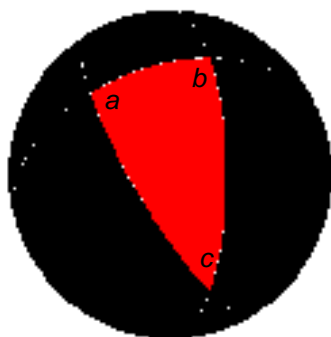
- ▶ Assumption 1: The sum of the angles of a triangle is less than  $180^\circ$ .
- ▶ Assumption 2: The sum of the angles of a triangle is greater than  $180^\circ$ .

Assumption 1 applies to hyperbolic geometry. Assume that the following figure is a concave disk. Note that the sides are curved inward, so the sum of the angles will be less than  $180^\circ$ .



$$\angle a + \angle b + \angle c < 180^\circ$$

Assumption 2 applies to spherical geometry. Assume that the following figure is a sphere. Lines are represented by curves on the great circles of the sphere. Note that they are curved outward, so the sum of the angles will be greater than  $180^\circ$ .



$$\angle a + \angle b + \angle c > 180^\circ$$

## Looking Deeper at Spherical Geometry (pp. 1 of 2)

### Basic information about spheres

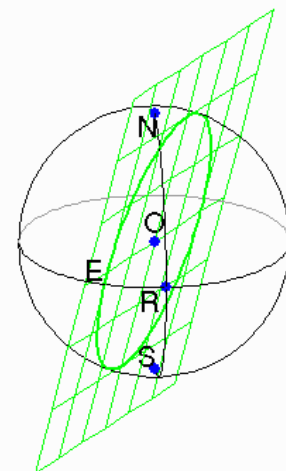
A *sphere* is a set of points in three-dimensional space equidistant from a point called the *center* of the sphere. The distance from the center to the points on the sphere is called the *radius* of the sphere. Notice that we are talking about the surface of a ball, and not the ball itself. The surface of the Earth is a good approximation to a sphere.

### Lines and spheres

Consider an arbitrary line and sphere. First, the line and the sphere can miss each other. That case is not very interesting. Secondly, the line can intersect the sphere in only one point. In that case the line is *tangent* to the sphere. The only other thing that can happen is that the line hits the sphere in precisely two points. The most interesting case is when the line passes through the center of the sphere. In this case the two points of intersection with the sphere are said to be *antipodal* points. The best-known example of antipodal points is the North and South Poles on the Earth.

### Planes, spheres, circles, and great circles

Consider a plane and a sphere. Again there are several things that can happen. In the uninteresting case, the plane and the sphere miss each other. If they do meet each other there are two possibilities. First, they can meet in a single point. In this case the plane is *tangent* to the sphere at the point of intersection. In the other case, the sphere and the plane meet in a circle. It is easy to see that the circle of intersection will be largest when the plane passes through the center of the sphere, as it does in the figure to the left. Such a circle is called a *great circle*. A geographic example of a great circle is the equator. The meridians of longitude on the Earth form great circles and always intersect at the North and South Poles. The latitudes on the Earth are small circles, except for the equator. They are always parallel.



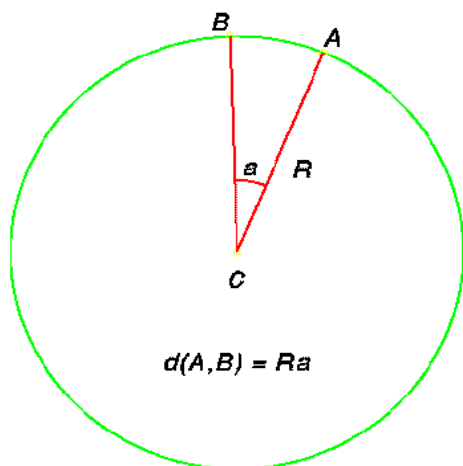
Great circles become more important when we realize that the shortest distance between two points on the sphere is along the segment of the great circle joining them. On any surface the curves that minimize the distance between points are called *geodesics*. Thus lines are the geodesics on the plane, and great circles fill that role on the sphere. A pretty good approximation to a great circle can be drawn through two points on a plastic sphere by holding a piece of string tight to the sphere at the two points in question. The tightness of the string has the effect of minimizing the length of the string, and therefore closely approximating a geodesic.

We now have the beginnings of geometry on the sphere. In Euclidean geometry the basic concepts are points and lines. On the sphere we have points, of course, but no lines as such. However, since the great circles are geodesics on the sphere, just as lines are in the plane, we should consider the great circles as replacements for lines. We can then compare the two geometries.

## Looking Deeper at Spherical Geometry (pp. 2 of 2)

### Incidence Relations on a Sphere

1. If point  $A$  and point  $B$  are two points on a sphere, which are not antipodal, then there is a unique great circle that contains both of them.
2. If point  $A$  and point  $B$  on the sphere are antipodal, then there are infinitely many great circles containing them. Why?
3. Two distinct great circles meet in exactly two antipodal points.



### Spherical distance

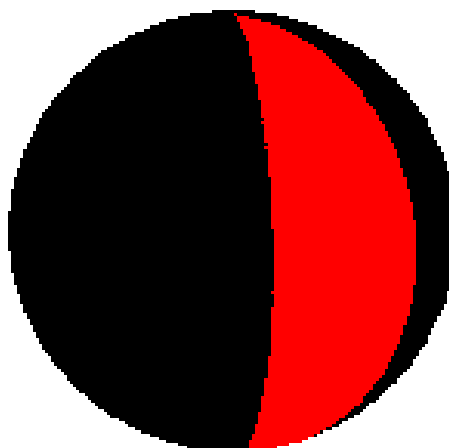
If  $A$  and  $B$  are two points on the sphere, then the distance between them is the distance along the great circle connecting them. Since this circle lies totally in a plane, we can figure this distance using the plane figure to our left. The length of  $\widehat{AB}$  is given by:

$d(A,B) = R a$ , where  $R$  is the radius of the sphere and  $a$  is the measure of the angle in radians.

### Lunes

In the plane the simplest polygon is the triangle. There are no interesting polygons with only two sides. This is not true on the sphere. Any two great circles meet in two antipodal points, and divide the sphere into four regions each of which has two sides which are segments of great circles. We will call such a region a *lune*, or a *biangle*. Lunes are pretty simple things. However there are two things we should notice about them.

- The vertices of a lune are antipodal points.
- The two angles of a lune are equal.



## Spherical Geometry (pp. 1 of 2) **KEY**

1. Compare the definition and dimensionality of a circle and a sphere?  
A circle is the set of points in a plane equidistant from a given point. It is a 2-dimensional figure, since it occurs in a plane. A sphere is the set of points in space equidistant from a given point. It is a 3-dimensional figure, since it occurs in space. The given point in each is the center, and the distance from the center to the circle or the sphere is called the radius.
2. What actually comprises the sphere?  
Only the surface area comprises the sphere.
3. What is a "line" on a sphere?  
A line on a sphere is a great circle of the sphere.
4. In Euclidean geometry, the intersection of two distinct lines is:  
A point or the entire line (two lines that coincide).
5. In spherical geometry, the intersection of two distinct lines is:  
Two points.
6. Explain why there are no parallel lines in spherical geometry.  
Since all lines on the sphere are great circles, they all intersect.
7. Are there parallel great circles on a sphere? To answer you will need to decide what *parallel* means on the sphere.  
No. Since parallel implies that lines do not intersect, it is not possible to have parallel great circles.
8. Explain the difference between a line segment in Euclidean geometry and in spherical geometry.  
In Euclidean Geometry, a line segment is a portion of a line. In Spherical Geometry, a line segment is a portion of a great circle.
9. In spherical geometry the shortest path between two points is not necessarily a unique path as in Euclidean geometry. When does this case occur?  
This situation occurs when the two points are such that they are the endpoints of a diameter of the sphere; therefore, the points are equidistant from each other along each semicircular arc of a great circle.



## Spherical Geometry (pp. 2 of 2) **KEY**

10. Write a conjecture about the sum of the measures of the angles of a triangle on a sphere.  
The sum of the angles of a triangle on a sphere are greater than  $180^\circ$ .
11. On the Earth what is the only latitude that forms a great circle?  
Equator
12. How do all the other latitudes compare to a great circle of the Earth?  
All other latitudes are smaller than the great circle.
13. Which longitude lines on the Earth are great circles?  
All longitude lines are great circles.
14. Where do all longitude lines on the Earth intersect?  
All longitude lines intersect at the North and South Pole
15. What two Euclidean geometry postulates do spherical and hyperbolic geometry contradict?
  - Parallel Postulate which states that through a given point not on a line, there is exactly one line passing through the point parallel to the given line.
  - The sum of the angles of a triangle equals  $180^\circ$ .
16. Explain how each of these postulates is contradicted by spherical and hyperbolic geometry.
  - Parallel Postulate which states that through a given point not on a line, there is exactly one line passing through the point parallel to the given line.
    - In spherical geometry all great circles intersect therefore there are no parallel "lines".
    - In hyperbolic geometry there are an infinite number of arcs that do not intersect a given arc.
  - The sum of the angles of a triangle equals  $180^\circ$ .
    - In spherical geometry the sum of the angles of a triangle is greater than  $180^\circ$ .
    - In hyperbolic geometry the sum of the angles of a triangle is less than  $180^\circ$ .

## Spherical Geometry (pp. 1 of 2)

1. Compare the definition and dimensionality of a circle and a sphere?
2. What actually comprises the sphere?
3. What is a “line” on a sphere?
4. In Euclidean geometry, the intersection of two distinct lines is:
5. In spherical geometry, the intersection of two distinct lines is:
6. Explain why there are no parallel lines in spherical geometry.
7. Are there parallel great circles on a sphere? To answer you will need to decide what *parallel* means on the sphere.
8. Explain the difference between a line segment in Euclidean geometry and in spherical geometry.
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## **Spherical Geometry** (pp. 2 of 2)

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## What Space Are You Living In? **KEY**

By now you realize that some Euclidean geometry notions are not always the same in other types of geometry or spaces. On #1 and 2 below compare and contrast the geometric concepts in both Euclidean and taxicab spaces. On #3 compare and contrast a concept of your choosing in both Euclidean and spherical and hyperbolic spaces.

1. The distance between two points in a plane.

Student responses may vary. Sample response given below.

The distance between two points  $(x_1, y_1)$  and  $(x_2, y_2)$  in taxicab geometry is the sum of the horizontal and vertical distances between the points given by the formula

$Path\ Distance = |x_1 - x_2| + |y_1 - y_2|$ . Euclidean distance for the points is the diagonal distance between the points given by the formula,  $Distance = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ . Generally speaking, the taxicab distance is more than the Euclidean distance. An exception to this occurs if the two points have the same x-coordinate or the same y-coordinate, then the distances are the same in each system with either formula yielding the correct distance.

2. The set of all points equidistant a given point in a plane.

Student responses may vary. Sample response given below.

In a Euclidean space, the set of all points equidistant a given point in a plane is a circle. In taxicab geometry, this “circle” is actually a Euclidean Square because distances are measure both horizontally and vertically, whereas in Euclidean geometry, distances can be measured diagonally between points.

3. A topic of your choosing for Euclidean vs. spherical and hyperbolic geometries.

Student responses may vary. Sample response given below.

Students may compare the Parallel Postulate to spherical having no parallel great circles (lines) and hyperbolic having infinitely many parallel arcs (lines).

Students may compare the sum of the angles equaling  $180^\circ$  to spherical where the sum of the angles is greater than  $180^\circ$  and hyperbolic where the sum of the angles is less than  $180^\circ$ .

## What Space Are You Living In?

By now you realize that some Euclidean geometry notions are not always the same in other types of geometry or spaces. On #1 and 2 below compare and contrast the geometric concepts in both Euclidean and taxicab spaces. On #3 compare and contrast a concept of your choosing in both Euclidean and spherical and hyperbolic spaces.

1. The distance between two points in a plane.
2. The set of all points equidistant a given point in a plane.
3. A topic of your choosing for Euclidean vs. spherical and hyperbolic geometries.